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A modified perturbation theory on the basis of spectrum generating algebras

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Abstract. A perturbation theory on the basis of spectrum generating algebras is developed. It offers several possibilities for splitting the algebraically formulated problem into an exact solvable one and a perturbation of it. Different conditions for the choice of the corresponding parameters are discussed. The simple charmonium potential $V=-Z/r+\lambda r$ and the potential of the quadratic Zeeman effect $V=-1/r+\gamma^2(x^2+y^2)$ are used as examples. Perturbation series with improved convergence properties are obtained.

1. Introduction

In the last few years many papers dealing with the calculation of large-order perturbation coefficients have been published. Since the classical Rayleigh-Schrödinger (RS) perturbation theory in general involves infinite sums over intermediate states (an important exception is the harmonic oscillator as the unperturbed problem), special methods to obtain exact expressions for the perturbation coefficients are needed. Some of these methods are the logarithmic perturbation theory (see e.g. Dolgov and Popov 1978, 1979, Aharonov and Au 1979, Eletsky *et al* 1981, Turbiner 1984), hypervirial perturbation treatments (e.g. Lai 1981, 1983), the algebraic perturbation theory (APT) using spectrum generating algebras (SGA) and dynamical algebras (e.g. Bednář 1973, Čižek and Vrscay 1977). The resulting perturbation series are in most cases non-convergent but only asymptotic (see, e.g., Killingbeck 1977). Therefore special techniques are needed to sum up the series such as the Padé approximation (Baker 1965) or the Borel and Padé-Borel summation (Avron 1981, Popov and Weinberg 1982).

Recently Feranchuk and Komarov (1982) developed a modified perturbation theory for the anharmonic oscillator $V=x^2+\lambda x^4$. Their operator method leads to very good results for lower orders. The operator method can easily be translated into the language of the harmonic oscillator o(2, 1)-SGA and in these terms generalised to other problems, which can be solved with SGA. This translation has been done in a letter by Gerry and Silverman (1983). With reference to the APT using SGA we call this method the modified algebraic perturbation theory (MAPT). Applying it to the Coulomb problem with a funnel-like perturbation potential $V_p=\lambda r$, they calculated the energy eigenvalue of the ground state up to the second order and obtained, for relatively small λ, very good results.

In the present paper we calculate higher orders of MAPT for $V=-1/r+\lambda r$. It turns out that the resulting series is not convergent but merely an asymptotic one (at least for not very small λ). By applying the Padé approximation we show that it has good

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summation properties. Furthermore we extend this method by taking advantage of the additional possibilities to divide the operator in the SGA approach into an unperturbed one and a perturbation. This considerably improves the convergence properties. MAPT using the SGA of the Coulomb problem is also applicable for pure polynomial potentials without a Coulomb term. As an example we use the potential $V = \lambda r$.

For the Coulomb problem the SGA can be embedded into the larger dynamical algebra o(4, 2). One representation of this algebra contains the components x_i of the position vector $\boldsymbol{r}$. Therefore non-central-symmetric perturbations can be expressed by representation operators. We demonstrate the applicability of the modified algebraic perturbation theory to non-central-symmetric perturbations by using the Zeeman problem.

2. The algebraic solution of the Coulomb problem and the algebraic perturbation theory (APT)

In this section we repeat briefly the known solution of the Coulomb problem with the help of the o(2, 1) SGA. For details see, for example, Cordero and Ghirardy (1972) or Bednář (1973). The operators

$$T_0 = r \cdot \boldsymbol{p} \cdot \boldsymbol{p}/4 + r, \qquad T_1 = r \cdot \boldsymbol{p} \cdot \boldsymbol{p}/4 - r, \qquad T_2 = \boldsymbol{r} \cdot \boldsymbol{p} - i \tag{2.1}$$

$$L_i = \varepsilon_{ijk} x_j p_k \qquad i, j, k = 1, 2, 3 \tag{2.2}$$

form a unitary representation of the algebra $o(2, 1) \oplus o(3)$. The corresponding scalar product is

$$\langle u, v \rangle := \left(u, \frac{1}{r} v \right) = \int u^*(\boldsymbol{r}) \frac{1}{r} v(\boldsymbol{r}) \, \mathrm{d}^3 \boldsymbol{r}. \tag{2.3}$$

This reducible representation decomposes into a direct sum of irreducible ones $D^+(\varphi) \otimes D_1$, with $\varphi = -l - 1$ and $l = 0, 1, \ldots$. A standard basis $|l, n, m\rangle$ for these irreducible representations can be chosen to have the properties

$$\begin{aligned} \boldsymbol{L}^2 |l, n, m\rangle &= l(l+1) |l, n, m\rangle \\ T_0 |l, n, m\rangle &= n |l, n, m\rangle \\ L_3 |l, n, m\rangle &= m |l, n, m\rangle \\ T_\pm |l, n, m\rangle &= [(n \mp l)(n \pm l \pm 1)]^{1/2} |l, n \pm 1, m\rangle \end{aligned} \tag{2.4}$$

with $T_\pm = T_1 \pm \mathrm{i} T_2$, $n = l + 1 + s$, $s = 0, 1, \ldots$, $m = -l, \ldots, +l$. The action of $L_\pm = L_1 \pm \mathrm{i} L_2$ on $|l, m, n\rangle$ is the usual one. In the following we write sometimes simply $|n\rangle$ instead of $|l, n, m\rangle$.

After multiplying the Schrödinger equation for the Coulomb problem by r it can be expressed by the operators (2.1) in the following way

$$\theta \psi_n = r(\tfrac{1}{2} \boldsymbol{p}^2 - r^{-1} - E_n) \psi_n = [(T_0 + T_1) - \tfrac{1}{2} E_n (T_0 - T_1) - 1] \psi_n. \tag{2.5}$$

Because of the relation

$$\exp(\mathrm{i} \vartheta T_2)(T_0 \pm T_1)^k \exp(-\mathrm{i} \vartheta T_2) = x^{\pm k} (T_0 \pm T_1)^k, \qquad x = \mathrm{e}^{-\vartheta} > 0 \qquad k \in N \tag{2.6}$$

by a tilting of (2.5) with $S_x = \exp(i\vartheta T_2)$ and choosing $\exp(-2\vartheta) = -\frac{1}{2}E$ the coefficient of T_1 can be made zero:

$$\tilde{\theta}\tilde{\psi}_n := (S_x \theta S_x^{-1}) S_x \psi_n = [(-2E_n)^{1/2} T_0 - 1]\tilde{\psi}_n = 0.$$

$\tilde{\psi}_n$ must be an eigenvector of T_0 and therefore we get as a result the well known energy spectrum $E_n = -1/(2n^2)$ $(n = l+1+s;\ l = 0, 1, \ldots;\ s = 0, 1, \ldots)$.

At this point it is easy to see how APT can be developed if the perturbation V_p is a polynomial one of r (see, e.g., Bednář 1973). The additional term $\lambda r V_p(r)$ in (2.5) can be expressed as a polynomial in $(T_0 - T_1)$. After 'tilting' by S, making a power series ansatz

$$E(\lambda) = \sum_{\nu=0}^{\infty} E_\nu \lambda^\nu, \qquad \tilde{\psi}(\lambda) = \sum_{\nu=0}^{\infty} \psi_\nu \lambda^\nu$$

and expanding ψ_ν in terms of the states $|n\rangle$ the coefficients E_ν and $\psi_\nu = \Sigma_n b_n^\nu |n\rangle$ can be calculated successively according to the known action of T_0 and T_1 on $|n\rangle$.

In contrast to the Rayleigh-Schrödinger (RS) perturbation theory on the basis of the original 'untilted' Schrödinger equation each term E_ν is directly and exactly calculable in the form of finite sums over intermediate states. No integration over the continuous spectrum has to be done. Nevertheless both the methods concern the same series expansion.

3. Modified algebraic perturbation theory for the Coulomb problem with polynomial perturbation. Case $V_p = \lambda r$

3.1. Theory

We consider as an example the problem $H = H_0 + V_p = \frac{1}{2}\boldsymbol{p}^2 - Z/r + \lambda r$. The generalisation of MAPT to other polynomial perturbations is straightforward. The starting point is also the tilted Schrödinger equation

$$\tilde{\theta}\tilde{\psi} = S_x r(\tfrac{1}{2}\boldsymbol{p}^2 - Z/r + \lambda r - E) S_x^{-1} \tilde{\psi} = 0. \tag{3.1}$$

A simple calculation using (2.1) and (2.6) yields an expression for $\tilde{\theta}$ which can be written as $\tilde{\theta} = \tilde{\theta}_0 + \tilde{\theta}_1$ with

$$\begin{aligned} \tilde{\theta}_0(E, x, P_i) &= (x - E/2x) T_0 - Z P_1 + V^d \\ \tilde{\theta}_1(E, x, P_i) &= (x + E/2x) T_1 + Z(P_1 - 1) + V^{d,p} + V^{\text{off}} \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} V^d &= \lambda_x (3 P_2 T_0^2 - P_3 \boldsymbol{L}^2)/2 \\ V^{d,p} &= \lambda_x [3(1 - P_2) T_0^2 - (1 - P_3)\boldsymbol{L}^2]/2 \\ V^{\text{off}} &= \lambda_x [(T_+^2 + T_-^2)/4 - T_0 T_1 - T_1 T_0]. \end{aligned} \tag{3.3}$$

The parameters x and P_i $(i = 1, 2, 3)$ can be chosen freely.

With the introduction of a formal perturbation parameter $\varkappa$ in $\tilde{\theta}$:

$$\tilde{\theta}(\varkappa, x, P_i)\tilde{\psi} = [\tilde{\theta}_0(x, P_i) + \varkappa \tilde{\theta}_1(x, P_i)]\tilde{\psi} = 0. \tag{3.4}$$

The expansion of $E(\varkappa)$ and $\tilde{\psi}(\varkappa)$:

$$\begin{aligned} E(\varkappa) &= E_0 + \varkappa E_1 + \ldots & E(1) &= E \\ \tilde{\psi}(\varkappa) &= \tilde{\psi}_0 + \varkappa \tilde{\psi}_1 + \ldots & \tilde{\psi}(1) &= \tilde{\psi} \end{aligned} \tag{3.5}$$

yields a system of equations for the determination of E_ν and $\tilde{\psi}_\nu$ by comparing the coefficients of equal powers of $\varkappa$:

$$0 = \tilde{\theta}_0(E_0, x, P_i)\tilde{\psi}_0 \tag{3.6a}$$

$$0 = \tilde{\theta}_0(E_0, x, P_i)\tilde{\psi}_\nu + \tilde{\theta}_1(0, x, P_i)\tilde{\psi}_{\nu-1} + \sum_{\mu=0}^{\nu-1} \frac{1}{2x}(-E_{\nu-\mu}T_0 + E_{\nu-\mu-1}T_1)\tilde{\psi}_\mu \qquad \nu \geqslant 1. \tag{3.6b}$$

It can be solved successively by expanding $\tilde{\psi}_\nu$ as $\tilde{\psi}_\nu = \Sigma_n b_n^\nu |n\rangle$. The splitting (3.4) is done in such a way that the new perturbation $\tilde{\theta}_1$ contains all off-diagonal (regarding the basis states $|n\rangle$) operators, i.e. T_1 and the off-diagonal part of V. The diagonal part of V and the constant term Z are shared among $\tilde{\theta}_0$ and $\tilde{\theta}_1$ with free parameters P_i. One vital point of MAPT is this degree of freedom in the splitting.

In APT the term $(x + E/2x)T_1$ is contained in the unperturbed operator. Therefore $x + E/2x$ must vanish in the zeroth order and x becomes $x = (-E_0/2)^{1/2}$. By contrast, in MAPT even x is a free parameter which can be adjusted together with P_i to improve the properties of the power series. From the following conditions the combinations a1-b1, a2-b1 and a2-b2 are used.

(a1) The parameters P_i are fixed in such a manner that $\tilde{\theta}_1$ contains only non-diagonal terms of $\tilde{\theta}$ (all $P_i = 1$), or

(a2) they are fixed by minimising $\Sigma_{\nu=0}^{\mu}(E_{N-\nu})^2$ for a large value of N and a small one for μ.

(b1) x is obtained from $E_0(x, P_i = 1) \to \min$.

(b2) x is determined by minimising $\Sigma_{\nu=0}^{\mu}(E_{N-\nu})^2$.

Condition b1 is a criterion for obtaining a zero-order wavefunction of the form $S_x^{-1}|n\rangle$ which minimises the energy expectation value. Because of (3.2), (2.4) and $\tilde{\psi}_0 \sim |n\rangle$:

$$\begin{aligned} 0 = \langle n|\tilde{\theta}_0(x, E_0, P_i = 1)|n\rangle &= \langle n|\tilde{\theta}_0 + \tilde{\theta}_1|n\rangle \\ &= \langle n|S_x r(H - E_0)S_x^{-1}|n\rangle \\ &= (S_x^{-1}n|(H - E_0)S_x^{-1}|n) \end{aligned}$$

and therefore

$$E_0(x) = (S_x^{-1}n|H|S_x^{-1}n)/(n|n),$$

so that $E_0(x)$ represents the energy expectation value.

The choice of P_i according to a1 is natural in the sense that one tries to put as much information as possible in the unperturbed problem. Gerry and Silverman (1983) used this criterion.

Condition a2-b2 is based on the idea that, for a series in which the contributions of consecutive terms $(E_{N-\mu}, \ldots, E_N)$ are small, the neighbouring terms should also have small values only. So, a series with improved properties is obtained. Although these arguments have no strong justification, the method gives satisfactory results, as shown in § 3.2.

An alternative to the series determined from the above conditions for fixed N is the computation of the sequence $E_{(0)}, E_{(1)}, \ldots, E_{(K)}, \ldots$

$$E_{(K)} = \sum_{\nu=0}^{K} E_\nu(x, P_i) \tag{3.7}$$

where x and P_i are determined at each order K with $N = K$. This can be qualified as an 'order dependent parameter adjustment'. For comparison with other methods it is useful to re-express the equation (3.4) in physical coordinates. With the abbreviations

$$H_0 = \tfrac{1}{2}\boldsymbol{p}^2 - ZP_1/xr$$

$$H_1 = Z(P_1 - 1)/xr + (\lambda/x^3)r$$

$$H_2 = \tfrac{1}{4}(\boldsymbol{p}\cdot\boldsymbol{p} - 4) + \frac{\lambda}{8x^3 r}[P_3 \boldsymbol{L}^2 - 3P_2(r - r\boldsymbol{p}^2/4)^2]$$

$$H_3 = \tfrac{1}{8}(\boldsymbol{p}\cdot\boldsymbol{p} - 4)$$

the eigenvalue problem

$$[H_0 + \varkappa H_1 + (\varkappa - 1)H_2]\psi = (E/x^2)[1 - (\varkappa - 1)H_3]\psi. \tag{3.8}$$

appears.

This complicated expression shows that, besides the operators $\boldsymbol{p}^2$ and $1/r$, r and powers of these operators also occur in the partition between the unperturbed problem and the perturbation. If we simplify this expression by setting $P_3 = P_2 = 0$, $Z = 1$ and cancelling the terms $(\boldsymbol{p}\cdot\boldsymbol{p} - 4)$ in H_1 and H_3 (this corresponds to including $(x + E/2x)T_1$ in $\tilde{\theta}_0$ instead of in $\tilde{\theta}_1$), we obtain

$$\left[\frac{\boldsymbol{p}^2}{2} - \frac{P_1}{r} + \varkappa\left(\frac{P_1 - 1}{r} + \lambda r\right)\right]\psi = E\psi. \tag{3.9}$$

This is exactly the partition investigated by Killingbeck (1981). As a condition for the free parameter P_1 he used the criterion

$$\partial E_{(N)}(P_1)/\partial P_1 = 0, \qquad E_{(N)} = \sum_{\nu=0}^{N} E_\nu \tag{3.10}$$

and calculated the elements E_ν using a hypervirial theorem.

It is interesting that the work of Killingbeck with (3.9) replaced by $E_N = 0$ can be shown to be a special case of the order dependent method (ODM) (Seznec and Zinn-Justin 1979). From scaling considerations of (3.9) it follows

$$E(\lambda, P_1, \varkappa) = [P_1 - \varkappa(P_1 - 1)]^2 E\left(\frac{\varkappa\lambda}{[P_1 + \varkappa(1 - P_1)]^3}\right), \tag{3.11}$$

$$E(\lambda, 1, 1) \equiv E(\lambda)$$

and by some algebraic manipulations

$$E(\lambda, P_1, 1) = \frac{1}{(1-\tau)^2}\sum_{\nu=0}^{\infty} \tau^\nu P^\nu(\rho), \tag{3.12}$$

$$\rho = \lambda/P_1(P_1 - 1), \qquad \tau = (P_1 - 1)/P_1.$$

Truncating now the series at order N and choosing ρ_N as a zero of P^N we obtain a sequence of partial sums $E_{(N)}$:

$$E_{(N)} = \frac{1}{(1-\tau)^2}\sum_{\nu=0}^{N} \tau^\nu P^\nu(\rho_N). \tag{3.13}$$

Because of $\lambda = \rho_N\tau/(1-\tau)^3$ the series (3.12) can be interpreted as a result of an adjustable change of the variable in the expansion of $E(\lambda)$. We have not investigated the convergence of (3.13) analytically, but the numerical results indicate the divergence of the sequence, at least for large values of λ. In the general case (3.7) a rescaling like (3.10) seems not to be possible so that it cannot be considered as a variant of ODM.

3.2. Results

For the parameter choice according to conditions a1-b1 and the states with $n = l+1$ we have calculated the first terms E_ν explicitly

$$\begin{aligned} E_0 &= 2x(x - Z/n) + x\lambda_x(2n+1) \\ E_2 &= -x\lambda_x^2(2n+1)/[2f(n+2)] \\ E_4 &= \frac{E_2\lambda_x^2}{f(n+2)}\left(\frac{4n+2}{f(n+1)} + \frac{24(n+1)}{f(n+3)} + \frac{3}{2}\frac{2n^2+5n+3}{f(n+4)}\right) + \frac{E_2^2(n+2)}{2xf(n+2)} \\ E_1 &= E_3 = 0 \end{aligned} \tag{3.14}$$

with

$$f(n') = (x - E_0/2x)n' - Z + \lambda_x(3n'^2 - l(l+1))/2$$

and $x > 0$ is a solution of

$$2nx^3 - Zx^2 = \lambda(2n^2+n)/8. \tag{3.15}$$

For the higher-order terms and the general case of P_i and x arbitrary numerical computations with a FORTRAN program had been carried out. Figure 1 represents the behaviour of some variants of the MAPT series for the $Z = 1$ ground state. In table 1 we compare the corresponding energy values with the results of a numerical integration of the Schrödinger equation. It can be seen from the figure that the properties of the series improve if we go from the conditions a1-b1 over a2-b1 to a2-b2. For $P_i = 1$ the series is strongly divergent. However, the value of the series truncated at the order of the smallest magnitude is accurate up to an error of less than one per cent. This order is always 4 for $\lambda \geqslant 1.0$. Perhaps this is connected with the fact that E_3 is exactly zero, according to (3.14).

Our tests of the condition a2 combined with b1 or b2 (minimising $\Sigma_{\nu=0}^{\mu}(E_{N-\nu})^2$) have shown that the obtained parameters P_i and x vary strongly with N and μ, if these numbers are small. Additionally, the convergence properties are unsatisfactory; it happens that the desired terms are small in magnitude and the neighbours are very large. Relatively stable results can be obtained when $N \geqslant 20$ and $\mu > 3$. Also the behaviour of the corresponding series at higher orders is good and improves with increasing N. For instance the term E_{100} for $\lambda = 100$, $\mu = 5$ and $N = 20$, 30, 40 and 50 has the values 5×10^{-3}, 4×10^{-5}, 3×10^{-6} and 5×10^{-7} respectively. Table 1 and figure 1 contain the case $N = 40$, $\mu = 5$. Because the series becomes an alternating one for higher orders, we have represented $E_{(99)} + E_{100}/2$ in table 1.

In the case a2-b2 the figure shows no evidence for a divergence of the series. We have also tested the order dependent parameter adjustment (3.7) up to order $N = 50$ with increasing accuracy (for $\lambda = 100$: $|E_{(50)} - E| = 1.4\times10^{-6}$). On the contrary, our numerical results of ODM (3.13) indicate a divergence of this method for large λ. For instance for $\lambda = 100$ and $N = 20$, 30, 40, 49 the error $|E_{(N)} - E|$ has the values 0.1, 0.3, 0.8 and 2.1, respectively. The results of all methods improve with decreasing λ.

One reason for the good properties of MAPT is perhaps the fact that the unperturbed operator in this method contains a term proportional to r which can reflect the behaviour of the problem for $r \to \infty$ correctly (see (3.8)).

One way to use the higher-order terms of a divergent series is to form Padé approximants (PA) (see, e.g., Baker 1965). We have calculated the [9, 9] and [9, 10] PA of our MAPT series (conditions a1-b1) and for comparison the approximants also

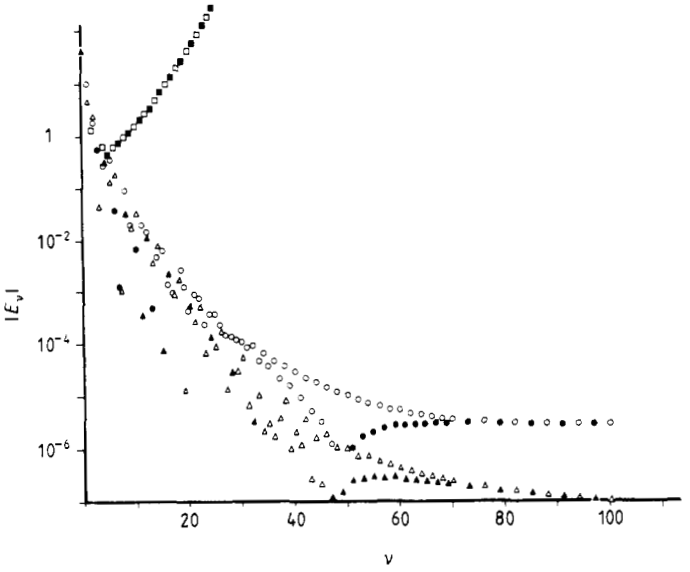

Figure 1. Magnitude of the contribution E_ν in the νth perturbation order of the MAPT. Potential $V=-1/r+\lambda r$, $\lambda=100$, ground state.

□ $E_\nu<0$
■ $E_\nu>0$
} condition a1-b1

○ $E_\nu<0$
● $E_\nu>0$
} condition a2-b1; $N=40$; $\mu=5$

△ $E_\nu<0$
▲ $E_\nu>0$
} condition a2-b2; $N=40$; $\mu=5$.

starting from the ordinary Rayleigh-Schrödinger (RS) perturbation series (Privman (1981) gives the coefficients up to the 20th order). The PA of our series give, in contrast to those of the ordinary PT, even for large values of λ the exact (numerical) results to several digits (see table 1). We must remark, however, that our series cannot be a Stieltjes series, because the $[N, N]$ and $[N, N+1]$ approximants are not lower and upper bounds of the exact value.

Table 2 contains the corresponding results for the excited levels $n=2$, $l=1$ and $n=2$, $l=0$. The perturbation coefficients of the RS series are obtained here from a modification of a program given by Killingbeck (1983).

The numbers show that MAPT provides energy values with a good precision also for high values of λ, i.e. for large perturbations of the original problem. This is connected with the fact that this method is applicable for $Z=0$ (pure polynomial potential without a Coulomb term), too. From (3.15) one obtains

$$x^3=(2n+1)\lambda/16$$

and from (3.14) for the ground-state energy up to the fourth order

$$E_{(4)}=\sum_{\nu=0}^{4}E_\nu=5.6564x^2=1.853\lambda^{2/3}.$$

This should be compared with the exact value (obtained from the tables of the Airy

Table 1. Comparison of several MAPT results and numerical values for the ground-state energy of the potential $V=-1/r+\lambda r$.

γ	0.5	1.0	10	100
numerical[a]	0.097 4205	0.577 921 35	6.143 4346	34.904 442
MAPT, $P_t=1$[b]	0.097 66	0.5764	6.129	34.835
MAPT+PA $P_t=1$[c] [9, 9]	0.097 420 56	0.577 9216	6.143 468	34.9049
MAPT+PA $P_t=1$[c] [9, 10]	0.097 420 55	0.577 9214	6.143 453	34.9047
RS+PA[d] [9, 9]	0.097 37	0.5766	5.20	9.97
RS+PA[d] [9, 10]	0.097 44	0.5788	6.90	65.70
MAPT, condition (a2), (b1)[e]	0.097 420 543 923 20 (-7.1×10^{-15})	0.577 921 351 961 (-4.1×10^{-12})	6.143 434 611 (-9.3×10^{-8})	34.904 4423 (-2.8×10^{-6})
MAPT, condition (a2), (b2)[f]	0.097 420 543 923 20 (-2.4×10^{-14})	0.577 921 351 9615 (-2.3×10^{-13})	6.143 434 609 02 (-8.1×10^{-10})	34.904 442 20 (-1.0×10^{-7})

[a] Values from a numerical integration of the radial Schrödinger equation.
[b] MAPT series truncated at the order of the smallest magnitude, conditions (a1) and (b1) of § 3.
[c] $[N, M]$ Padé approximation performed with the MAPT series, condition (a1) and (b1).
[d] $[N, M]$ Padé approximation performed with the RS series.
[e] $E_{(99)}+E_{100}/2$ of the MAPT series, condition (a2) and (b1) $N=40$, $\mu=5$, the value in parenthesis is E_{100}.
[f] See [e], however condition (b1) is replaced by (b2).

Table 2. MAPT results for excited states of the potential $V=-1/r+\lambda r$.

	$n=2,\ l=1$		$n=2,\ l=0$	
λ	10	100	10	100
numerical[a]	10.946 45	54.435 96	13.418 39	66.436 89
MAPT, $P_t=1$[b]	10.92	54.33	12.54	64.75
MAPT+PA $P_t=1$[c] [9, 9]	10.946 489	54.436 198	13.458	66.58
MAPT+PA $P_t=1$[c] [14, 14]	10.946 459	54.435 972	13.409	66.44
RS+PA[d] [9, 9]	0.63	0.64	2.82	3.08
RS+PA[d] [8, 9]	42.52	425.88	22.46	223.15
MAPT, condition (a2), (b1)[e]	10.946 459 23 (-1.5×10^{-7})	54.435 9682 (-5.0×10^{-6})	13.418 3952 (-1.2×10^{-7})	66.436 927 (-2.5×10^{-6})
MAPT, condition (a2), (b2)[f]	10.946 459 2335 (-2.4×10^{-8})	54.435 968 255 (-4.1×10^{-9})	13.418 3945 (-7.1×10^{-8})	66.436 922 (-2.0×10^{-6})

[a–f] As in table 1.

function, see Abramowitz and Stegun (1965)) of

$$E = 1.8558\lambda^{2/3}$$

i.e. the relative error is about 2‰. It is remarkable that our expansion yields the exact scaling behaviour $E \sim \lambda^{2/3}$ in each order as an inspection of (3.6) shows.

4. Extension to non-central-symmetric perturbations: the Zeeman problem

There has been a growing interest in the past few years in the Zeeman problem of high magnetic fields (see, e.g., Garstang 1977). One of the main motivations came from astrophysics (high magnetic fields on neutron star surfaces) and the other from solid state physics (large effective magnetic fields for excitons in solids). Therefore we choose for a non-central-symmetric example of MAPT the Coulomb problem in a magnetic field:

$$H = \tfrac{1}{2}\boldsymbol{p}^2 - r^{-1} + \tfrac{1}{8}\gamma^2(x^2+y^2). \tag{4.1}$$

Here a trivial term proportional to L_z is omitted.

In order to express the Hamiltonian (4.1) by representation operators, we must extend the representation (2.1) and (2.2) of $o(2,1)\oplus o(3)$ to one of the larger algebra $o(4,2)$. Additional operators are (see, e.g., Bednář 1973):

$$\begin{aligned} \boldsymbol{A} &= \tfrac{1}{2}[\tfrac{1}{2}\boldsymbol{r}\boldsymbol{p}^2 + \mathrm{i}\boldsymbol{p} - (\boldsymbol{r}\boldsymbol{p})\boldsymbol{p} + 2\boldsymbol{r}] \\ \boldsymbol{B} &= \tfrac{1}{2}[\tfrac{1}{2}\boldsymbol{r}\boldsymbol{p}^2 + \mathrm{i}\boldsymbol{p} - (\boldsymbol{r}\boldsymbol{p})\boldsymbol{p} - 2\boldsymbol{r}] \\ \Gamma &= -\boldsymbol{r}\boldsymbol{p}. \end{aligned} \tag{4.2}$$

Together with the operators (2.1) and (2.2) they form an irreducible, unitary representation of $o(4,2)$ with the representation space generated by the states $|n, l, m\rangle$ ($n = 0, 1, \ldots$; $l = 0, 1, \ldots, n-1$; $m = -l, \ldots, l$; see (2.4)) and the scalar product (2.3). The action of the operators (4.2) on the states are known as well as their commutation relations. From one of the commutation relations:

$$[T_2, B_3 \pm A_3] = \pm\mathrm{i}(B_3 \pm A_3)$$

follows

$$S_x(B_3 \pm A_3)^k S_x^{-1} = x^{\pm k}(B_3 \pm A_3)^k.$$

This is analogous to (2.6).

Now we can write the tilted Schrödinger equation $\tilde{\theta}\tilde{\psi} = 0$ with

$$\begin{aligned} \tilde{\theta} &= S_x r(H-E)S_x^{-1} \\ &= x(T_0+T_1) - \frac{E}{2x}(T_0-T_1) - 1 + \frac{\gamma^2}{64x^2}[(T_0-T_1)^3 - (T_0-T_1)(B_3-A_3)^2]. \end{aligned}$$

In this paper we only consider the ground state and a partition of $\tilde{\theta}$ in a new unperturbed operator $\tilde{\theta}_0$ and a new perturbed one $\tilde{\theta}_1$ according to the conditions a1 and b1 in § 3, i.e. $\tilde{\theta}_1$ contains only non-diagonal (on the states $|n, l, m\rangle$) operators. Excited states and other partitions of $\tilde{\theta}$ are treated in a further paper.

Table 3. Energy values of the Zeeman ground state for various γ. The abbreviations are the same as in table 1.

γ		0.5	1	10	100
Literature	(i)				46.210
	(ii)		−0.331 16		
	(iii)	−0.447 237	−0.331 66	3.2287	46.203
	(iv)	−0.447 210 538	−0.331 168 896	3.2522	46.210
MAPT E_0		−0.443 36	−0.3087	4.273	62.17
E_0+E_2		−0.447 73	−0.3339	3.368	50.21
MAPT+PA	[4, 4]	−0.447 2103	−0.331 145	3.268	47.16
	[8, 8]	−0.447 2105	−0.331 165	3.259	46.75
RS+PA		−0.4473	−0.3400	0.065	0.08

(i) Simola and Virtano (1978), numerical self-consistent calculation.
(ii) Praddaude (1972), diagonalisation of H in a basis with cylindrical symmetry.
(iii) Galindo and Pascual (1976), special variant of PA based on the RS series.
(iv) Le Guillou and Zinn-Justin (1983), summation of the RS series with order dependent mapping.

Table 3 contains our results together with probably the best available values from some other authors. Since our series diverges rapidly for large values of γ we have only listed E_0 and E_0+E_2 ($E_1=0$). The higher terms are used in forming PA. For comparison we also list the PA formed with the ordinary perturbation series (the coefficients of which we have from Avron *et al* (1979)).

It can be seen that the PA from the MAPT series, in contrast to those of the ordinary series, also gives reasonable results for large γ. They are less accurate than those of Le Guillou and Zinn-Justin (1983) and for $\gamma=100$ also less accurate than those of Galindo and Pascual (1976). This is connected with the fact that Galindo and Pascual have used the asymptotic behaviour of $E(\gamma)$ for $\gamma\to\infty$ as an additional input. We have not aspired to attain extreme accuracy; rather we have tried to demonstrate the applicability of our method. Thus we calculated our series only up to the 18th order contrary to Le Guillou and Zinn-Justin who used the first 62 coefficients of the perturbation series. Moreover, we have not yet used the additional freedom for the splitting of $\tilde{\theta}$ in $\tilde{\theta}_0$ and $\tilde{\theta}_1$ as has been done in the case of central-symmetric perturbations. A further improvement of the series should be reached by taking advantage of the additional freedom.

5. Summary and conclusions

Using the Coulomb problem with additional potentials $V_p\sim r$ (simple charmonium potential) and $V_p\sim x^2+y^2$ (quadratic Zeeman effect) the following have been demonstrated.

(i) If a quantum problem is represented by o(2, 1)-SGA operators or operators of a larger algebra containing o(2, 1), various splittings of the eigenvalue problem $\tilde{\theta}\tilde{\psi}=0$ in an unperturbed and a perturbed one become visible. They are expressed by an additional parameter P_i and also depend on the tilting parameter x.

(ii) The corresponding perturbation theory can be explained by purely algebraic operations and therefore can be realised easily on a computer. For this reason very high orders of the perturbation series are calculable.

(iii) By an appropriate choice of the tilting parameter x and the splitting parameter P_i the method yields good results for higher orders. As a rule the resulting series is not convergent but is only an asymptotic one.

(iv) If the perturbation series itself is too unsatisfactory (e.g. for $P_i = 1$) PA improves the results essentially also in those cases in which the PA of the ordinary perturbation series fails.

The question of whether there exist parameters P_i and x for which the series is convergent (the letter of Fernández and Castro (1982) indicates this) is open. An analysis analogous to that by Halliday and Suranyi (1979) for the anharmonic oscillator (in the context of creation and annihilation operators) should solve this problem.

The MAPT is applicable to other polynomial perturbations of the Coulomb problem. Examples of renewed interest are the Stark effect and crossed electric and magnetic fields. The potential $V = e^{-\lambda r}$ as an example for a non-polynomial problem can be handled by expanding it in a Taylor series and using λ as a perturbation parameter.

On the other hand, we can start from other problems exactly solvable with SGA such as the relativistic Coulomb problem of spinless particles (Klein-Gordon equation) or the dyonium (Barut and Bornzin 1971). Polynomial perturbations of these problems can also be treated along the same lines.

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